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PARALLEL π-VECTOR FIELDS AND ENERGY β-CHANGE

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The present paper deals with an *intrinsic* investigation of the notion of a parallel π-vector field on the pullback bundle of a Finsler manifold (M, L). The effect of the existence of a parallel π-vector field on some important special Finsler spaces is studied. An intrinsic investigation of a particular β-change, namely the energy β-change $(\widetilde{L}^2(x,y) = L^2(x,y) + B^2(x,y)$ *with* $B := g(\overline{\xi}(x), \overline{\eta})$; $\overline{\xi}(x)$ being a parallel π-vector field), is established. The relation between the two Barthel connections Γ and $\widetilde{\Gamma}$, corresponding to this change, is found. This relation, together with the fact that the Cartan and the Barthel connections have the same horizontal and vertical projectors, enable us to study the energy β-change of the fundamental linear connection in Finsler geometry: The Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection. Moreover, the change of their curvature tensors is concluded.

It should be pointed out that the present work is formulated in a prospective modern coordinate-free form.

Keywords: Special Finsler space; energy β-change; parallel π-vector field; canonical spray; Barthel connection; Cartan connection; Berwald connection; Chern connection; Hashiguchi connection.

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0. Introduction

Studying Finsler geometry, however, one encounters substantial difficulties trying to seek analogues of classical global, or sometimes even local, results of Riemannian geometry. These difficulties arise mainly from the fact that in Finsler geometry all geometric objects depend not only on positional coordinates, as in Riemannian geometry, but also on directional arguments.

In Riemannian geometry, there is a canonical linear connection on the manifold M, whereas in Finsler geometry there is a corresponding canonical linear connection due to Cartan. However, this is not a connection on M but is a connection on $T(\mathcal{T}M)$, the tangent bundle of $\mathcal{T}M$, or on $\pi^{-1}(TM)$, the pullback of the tangent bundle TM by $\pi : \mathcal{T}M \to M$.

The concept of a parallel vector field in Riemannian geometry had been studied by many authors. On the other hand, the notion of a parallel vector field in Finsler geometry had been studied *locally* by Kitayama [7] and others.

In this paper, we study *intrinsically* the notion of a parallel π-vector field on the pullback bundle $\pi^{-1}(TM)$ of a Finsler manifold (M, L). Some properties of parallel π-vector fields are discussed. These properties, in turn, play a key role in obtaining other interesting results. The effect of the existence of a parallel π-vector field on some important special Finsler spaces is investigated.

The infinitesimal transformations (changes) in Finsler geometry are important, not only in differential geometry, but also in application to other branches of science, especially in the process of geometrization of physical theories [9, 10].[a] For this reason and as application of a parallel π-vector field, we investigate intrinsically a particular β-change, which will be referred to as an energy β-change[b]:

$$\widetilde{L}^2(x,y) = L^2(x,y) + B^2(x,y),$$

where (M, L) is a Finsler manifold admitting a parallel π-vector field $\overline{\xi}(x)$ and $B := g(\overline{\xi}(x), \overline{\eta})$; $\overline{\eta}$ being the fundamental π-vector field. Moreover, the relation between the two Barthel connections Γ and $\widetilde{\Gamma}$, corresponding to this change, is obtained. This relation, together with the fact that the Cartan and the Barthel connections have the same horizontal and vertical projectors, enable us to study the energy β-change of the fundamental linear connections on the pullback bundle of a Finsler manifold, namely, the Cartan connection, the Berwald connection, the Chern connection and the Hashiguchi connection. Moreover, the change of their curvature tensors is concluded.

Finally, it should be pointed out that the present work is formulated in a prospective modern coordinate-free form.

1. Notation and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [1, 3] and [11]. We assume, unless otherwise stated, that all geometric objects treated are of class C^∞. The following notation will be used throughout this paper:

M : A real paracompact differentiable manifold of finite dimension n and of class C^∞,

$\mathfrak{F}(M)$: The $\mathbb{R}$-algebra of differentiable functions on M,

[a]In 1941, Randers published his paper "On an asymmetrical metric in the four-space of general relativity". In this paper, Randers considered the simplest possible asymmetrical generalization of a Riemannian metric. Adding a one-form to the existing Riemannian structure, he was the first to introduce a special Finsler space. This space — which became known in the literature as a Randers space — proved to be mathematically and physically very important. It was one of the first attempts to study a physical theory in the wider context of Finsler geometry, although Randers was not aware that the geometry he used was a special type of Finsler geometry.

[b]Energy β-change by using a concurrent π-vector field is studied in [15].

$$\begin{aligned}
\mathfrak{X}(M) &: \text{The } \mathfrak{F}(M)\text{-module of vector fields on } M,\\
\pi_M : TM \to M &: \text{The tangent bundle of } M,\\
\pi_M^* : T^*M \to M &: \text{The cotangent bundle of } M,\\
\pi : \mathcal{T}M \to M &: \text{The sub-bundle of nonzero vectors tangent to } M,\\
V(TM) &: \text{The vertical sub-bundle of the bundle } TTM,\\
P : \pi^{-1}(TM) \to \mathcal{T}M &: \text{The pullback of the tangent bundle } TM \text{ by } \pi,\\
P^* : \pi^{-1}(T^*M) \to \mathcal{T}M &: \text{The pullback of the cotangent bundle } T^*M \text{ by } \pi,\\
\mathfrak{X}(\pi(M)) &: \text{The } \mathfrak{F}(\mathcal{T}M)\text{-module of differentiable}\\
&\qquad \text{sections of } \pi^{-1}(TM),\\
\mathfrak{X}^*(\pi(M)) &: \text{The } \mathfrak{F}(\mathcal{T}M)\text{-module of differentiable}\\
&\qquad \text{sections of } \pi^{-1}(T^*M),\\
i_X &: \text{The interior product with respect to } X \in \mathfrak{X}(M),\\
df &: \text{The exterior derivative of } f \in \mathfrak{F}(M),\\
d_L &:= [i_L, d],\ i_L \text{ being the interior derivative with}\\
&\qquad \text{respect to a vector form } L.
\end{aligned}$$

Elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles, relating the tangent bundle $T(\mathcal{T}M)$ and the pullback bundle $\pi^{-1}(TM)$:

$$0 \to \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \to 0,$$

where the bundle morphisms ρ and γ are defined respectively by $\rho := (\pi_{\mathcal{T}M}, d\pi)$ and $\gamma(u, v) := j_u(v)$, where j_u is the natural isomorphism $j_u : T_{\pi_M(v)}M \to T_u(T_{\pi_M(v)}M)$. The vector one-form J on TM defined by $J := \gamma \circ \rho$ is called the natural almost tangent structure of TM. The vertical vector field $\mathcal{C}$ on TM defined by $\mathcal{C} := \gamma \circ \overline{\eta}$ is called the fundamental or the canonical (Liouville) vector field.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. The map

$$K : T\mathcal{T}M \to \pi^{-1}(TM) : X \mapsto D_X\overline{\eta}$$

is called the connection map or the deflection map associated with D. A tangent vector $X \in T_u(\mathcal{T}M)$ is said to be horizontal if $K(X) = 0$. The vector space $H_u(\mathcal{T}M)$ of the horizontal vectors at $u \in \mathcal{T}M$ is called the horizontal space of M at u. The connection D is said to be regular if

$$T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \quad \forall\, u \in \mathcal{T}M. \tag{1.1}$$

If M is endowed with a regular connection D, then the maps

$$\begin{aligned}
\gamma &: \pi^{-1}(TM) \to V(\mathcal{T}M),\\
\rho|_{H(\mathcal{T}M)} &: H(\mathcal{T}M) \to \pi^{-1}(TM),\\
K|_{V(\mathcal{T}M)} &: V(\mathcal{T}M) \to \pi^{-1}(TM)
\end{aligned}$$

are vector bundle isomorphisms. Let $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$, called the horizontal map associated with D, then

$$\rho \circ \beta = \mathrm{id}_{\pi^{-1}(TM)}, \quad \beta \circ \rho = \begin{cases} \mathrm{id}_{H(\mathcal{T}M)} & \text{on } H(\mathcal{T}M) \\ 0 & \text{on } V(\mathcal{T}M). \end{cases} \tag{1.2}$$

For a regular connection D, the horizontal and vertical covariant derivatives $\overset{1}{D}$ and $\overset{2}{D}$ are defined, for a vector $(1)\pi$-form A, for example, by

$$(\overset{1}{D}A)(\overline{X}, \overline{Y}) := (D_{\beta\overline{X}}A)(\overline{Y}), \quad (\overset{2}{D}A)(\overline{X}, \overline{Y}) := (D_{\gamma\overline{X}}A)(\overline{Y}).$$

The (classical) torsion tensor $\mathbf{T}$ of the connection D is given by

$$\mathbf{T}(X, Y) = D_X\rho Y - D_Y\rho X - \rho[X, Y] \quad \forall\, X, Y \in \mathfrak{X}(\mathcal{T}M),$$

from which the horizontal or (h)h-torsion tensor Q and the mixed or (h)hv-torsion tensor T are defined respectively by

$$Q(\overline{X}, \overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) = \mathbf{T}(\gamma\overline{X}, \beta\overline{Y}) \quad \forall\, \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

The (classical) curvature tensor $\mathbf{K}$ of the connection D is given by

$$\mathbf{K}(X, Y)\rho Z = -D_X D_Y \rho Z + D_Y D_X \rho Z + D_{[X,Y]}\rho Z \quad \forall\, X, Y, Z \in \mathfrak{X}(\mathcal{T}M),$$

from which the horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors, denoted by R, P and S respectively, are defined by

$$R(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}, \gamma\overline{Y})\overline{Z},$$
$$S(\overline{X}, \overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X}, \gamma\overline{Y})\overline{Z}.$$

The contracted curvature tensors $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$, also known as the (v)h-, (v)hv- and (v)v-torsion tensors, are defined by

$$\widehat{R}(\overline{X}, \overline{Y}) = R(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X}, \overline{Y}) = P(\overline{X}, \overline{Y})\overline{\eta}, \quad \widehat{S}(\overline{X}, \overline{Y}) = S(\overline{X}, \overline{Y})\overline{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(R(\overline{X}, \overline{Y})\overline{Z}, \overline{W}), \ldots, S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := g(S(\overline{X}, \overline{Y})\overline{Z}, \overline{W}). \tag{1.3}$$

On a Finsler manifold (M, L), there are *canonically* associated four linear connections on $\pi^{-1}(TM)$ [16]: The Cartan connection ∇, the Chern (Rund) connection D^c, the Hashiguchi connection D^* and the Berwald connection D°. Each of these connections is regular with (h)hv-torsion T satisfying $T(\overline{X}, \overline{\eta}) = 0$. The following theorem guarantees the existence and uniqueness of the Cartan connection on the pullback bundle.

Theorem 1.1 ([13]). *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection ∇ on $\pi^{-1}(TM)$*

such that

(i) ∇ *is metric*: $\nabla g = 0$.
(ii) *The* (h)h-*torsion of* ∇ *vanishes*: $Q = 0$.
(iii) *The* (h)hv-*torsion* T *of* ∇ *satisfies*: $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$.

Definition 1.2. Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. We define *:*

$$\begin{aligned}
\ell(\overline{X}) &:= L^{-1} g(\overline{X},\overline{\eta}). \\
\hbar &:= g - \ell \otimes \ell : \text{The angular metric tensor.} \\
T(\overline{X},\overline{Y},\overline{Z}) &:= g(T(\overline{X},\overline{Y}),\overline{Z}) : \text{The Cartan tensor.} \\
C(\overline{X}) &:= \mathrm{Tr}\{\overline{Y} \mapsto T(\overline{X},\overline{Y})\} : \text{The contracted torsion.} \\
g(\overline{C},\overline{X}) &:= C(\overline{X}) : \overline{C} \text{ is the } \pi\text{-vector field associated with the } \pi\text{-form } C. \\
\mathrm{Ric}^v(\overline{X},\overline{Y}) &:= \mathrm{Tr}\{\overline{Z} \mapsto S(\overline{X},\overline{Z})\overline{Y}\} : \text{The vertical Ricci tensor.} \\
g(\mathrm{Ric}^v_0(\overline{X}),\overline{Y}) &:= \mathrm{Ric}^v(\overline{X},\overline{Y}) : \text{The vertical Ricci map } \mathrm{Ric}^v_0. \\
Sc^v &:= \mathrm{Tr}\{\overline{X} \mapsto \mathrm{Ric}^v_0(\overline{X})\} : \text{The vertical scalar curvature.}
\end{aligned}$$

Deicke theorem [2] can be formulated globally as follows:

Lemma 1.3. *Let (M, L) be a Finsler manifold. The following assertions are equivalent*:

(i) (M, L) *is Riemannian.*
(ii) *The* (h)hv-*torsion tensor T vanishes.*
(iii) *The π-form C vanishes.*

The following two results [16] give an explicit expression for each of the Berwald, Chern and Hashiguchi connections in terms of the Cartan connection ∇.

Theorem 1.4. *The Chern connection D^c is given, in terms of Cartan connection, by*

$$D^c_X\overline{Y} = \nabla_X\overline{Y} - T(KX,\overline{Y}) = D^\circ_X\overline{Y} - \widehat{P}(\rho X,\overline{Y}).$$

In particular, we have

(i) $D^c_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X},\overline{Y}) = D^\circ_{\gamma\overline{X}}\overline{Y}$.
(ii) $D^c_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} = D^\circ_{\beta\overline{X}}\overline{Y} - \widehat{P}(\overline{X},\overline{Y})$.

Theorem 1.5. *The Hashiguchi connection D^* is given, in terms of Cartan connection, by*

$$D^*_X\overline{Y} = \nabla_X\overline{Y} + \widehat{P}(\rho X,\overline{Y}) = D^\circ_X\overline{Y} + T(KX,\overline{Y}).$$

In particular, we have

(i) $D^*_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y} = D^\circ_{\gamma\overline{X}}\overline{Y} + T(\overline{X},\overline{Y})$.
(ii) $D^*_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y} + \widehat{P}(\overline{X},\overline{Y}) = D^\circ_{\beta\overline{X}}\overline{Y}$.

We terminate this section by some concepts and results concerning the Klein–Grifone approach to intrinsic Finsler geometry. For more details, we refer to [5, 6] and [8].

A semispray is a vector field X on TM, C^∞ on $\mathcal{T}M$, C^1 on TM, such that $\rho \circ X = \overline{\eta}$. A semispray X which is homogeneous of degree two in the directional argument ($[\mathcal{C}, X] = X$) is called a spray.

Proposition 1.6 ([8]). *Let (M, L) be a Finsler manifold. The vector field G on TM defined by $i_G \Omega = -dE$ is a spray, where $E := \frac{1}{2}L^2$ is the energy function and $\Omega := dd_J E$. Such a spray is called the canonical spray.*

A nonlinear connection on M is a vector one-form Γ on TM, C^∞ on $\mathcal{T}M$, C^0 on TM, such that

$$J\Gamma = J, \quad \Gamma J = -J.$$

The horizontal and vertical projectors h_Γ and v_Γ associated with Γ are defined by $h_\Gamma := \frac{1}{2}(I + \Gamma)$ and $v_\Gamma := \frac{1}{2}(I - \Gamma)$. To each nonlinear connection Γ there is associated a semispray S defined by $S = h_\Gamma S'$, where S' is an arbitrary semispray. A nonlinear connection Γ is homogeneous if $[\mathcal{C}, \Gamma] = 0$. The torsion of a nonlinear connection Γ is the vector two-forms t on TM defined by $t := \frac{1}{2}[J, \Gamma]$. The curvature of Γ is the vector two-forms $\mathfrak{R}$ on TM defined by $\mathfrak{R} := -\frac{1}{2}[h_\Gamma, h_\Gamma]$. A nonlinear connection Γ is said to be conservative if $d_{h_\Gamma} E = 0$.

Theorem 1.7 ([6]). *On a Finsler manifold (M, L), there exists a unique conservative homogeneous nonlinear connection with zero torsion. It is given by*:

$$\Gamma = [J, G],$$

where G is the canonical spray.

Such a nonlinear connection is called the canonical connection, the Barthel connection or the Cartan nonlinear connection associated with (M, L).

It should be noted that the semispray associated with the Barthel connection is a spray, which is the canonical spray.

2. Finsler Spaces Admitting a Parallel π-Vector Field

In this section, we introduce and investigate **intrinsically** the notion of a parallel π-vector field in Finsler geometry. The properties of parallel π-vector fields are obtained.

In what follows ∇ will denote the Cartan connection associated with a Finsler manifold (M, L) and S, P and R will denote the three curvature tensors of ∇.

Definition 2.1. Let (M, L) be a Finsler manifold. Let ∇ be the Cartan connection in $\pi^{-1}(TM)$. A π-tensor field ω is called

(i) ∇-horizontally parallel if $\overset{1}{\nabla}\omega = 0$,

(ii) ∇-vertically parallel if $\overset{2}{\nabla}\omega = 0$,
(iii) ∇-parallel if both (i) and (ii) satisfy, i.e. $\nabla\omega = 0$.

Remark 2.2. The set of all ∇-parallel π-tensor fields of the same type form a vector space over $\mathbb{R}$.

Lemma 2.3. *Let (M, L) be a Finsler manifold. If $\overline{\xi}(x, y) \in \mathfrak{X}(\pi(M))$ is a π-vector field and $\alpha \in \mathfrak{X}^*(\pi(M))$ is the π-form associated with $\overline{\xi}$ under the duality defined by the metric $g : \alpha = i_{\overline{\xi}}\, g$, then the π-form α is ∇-parallel, if and only if, $\overline{\xi}$ is ∇-parallel.*

Proof. The proof follows from the expression of $\nabla\alpha$, taking into account the definition of α and the fact that $\nabla g = 0$. □

Now, we have the following

Proposition 2.4. *Let $\overline{\xi} \in \mathfrak{X}(\pi(M))$ be a ∇-parallel π-vector field on (M, L). For the v-curvature tensor S, the following relations hold*:

(i) $S(\overline{X}, \overline{Y})\,\overline{\xi} = 0, \quad S(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$
(ii) $(\overset{1}{\nabla} S)(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$
(iii) $(\overset{2}{\nabla} S)(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$

For the hv-*curvature tensor P, the following relations hold*:

(iv) $P(\overline{X}, \overline{Y})\,\overline{\xi} = 0, \quad P(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$
(v) $(\overset{1}{\nabla} S)(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$
(vi) $(\overset{2}{\nabla} S)(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$

For the h-*curvature tensor R, the following relations hold*:

(vii) $R(\overline{X}, \overline{Y})\,\overline{\xi} = 0, \quad R(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$
(viii) $(\overset{1}{\nabla} R)(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$
(ix) $(\overset{2}{\nabla} R)(\overline{X}, \overline{Y}, \overline{Z}, \overline{\xi}) = 0.$

Proof. The proof is clear and we omit it. □

Lemma 2.5. *Let (M, L) be a Finsler manifold and D° the Berwald connection on $\pi^{-1}(TM)$. Then, we have*

(i) *A π-vector field $\overline{Y} \in \mathfrak{X}(\pi(M))$ is independent of the directional argument y, if and only if, $D^\circ_{\gamma\overline{X}}\overline{Y} = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$.*
(ii) *A scalar (vector) π-form ω is independent of the directional argument y, if and only if, $D^\circ_{\gamma\overline{X}}\,\omega = 0$ for all $\overline{X} \in \mathfrak{X}(\pi(M))$.*

Proposition 2.6. *Let (M, L) be a Finsler manifold. Let $\overline{\xi}(x) \in \mathfrak{X}(\pi(M))$ be a ∇-parallel π-vector field. For every $\overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M))$, we have*

(i) $T(\overline{X}, \overline{\xi}) = T(\overline{\xi}, \overline{X}) = 0$.
(ii) $\widehat{P}(\overline{\xi}, \overline{X}) = \widehat{P}(\overline{X}, \overline{\xi}) = 0$.
(iii) $P(\overline{X}, \overline{\xi})\overline{Y} = P(\overline{\xi}, \overline{X})\overline{Y} = 0$.

Proof. (i) The proof follows from the fact that $\overline{\xi}(x) \in \mathfrak{X}(\pi(M))$ is a ∇-parallel π-vector field, taking into account Theorem 1.4(i) and Lemma 2.5.

(ii) Follows from the identity $\widehat{P}(\overline{X}, \overline{Y}) = (\nabla_{\beta\overline{\eta}}T)(\overline{X}, \overline{Y})$ [17], making use of (i) and the fact that $T(\overline{X}, \overline{\eta}) = 0$.

(iii) We have [17]

$$\begin{aligned} P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) &= g((\nabla_{\beta\overline{Z}}T)(\overline{Y}, \overline{X}), \overline{W}) - g((\nabla_{\beta\overline{W}}T)(\overline{Y}, \overline{X}), \overline{Z}) \\ &\quad - g(T(\overline{X}, \overline{W}), \widehat{P}(\overline{Z}, \overline{Y})) + g(T(\overline{X}, \overline{Z}), \widehat{P}(\overline{W}, \overline{Y})). \end{aligned} \tag{2.1}$$

From which, by setting $\overline{Y} = \overline{\xi}$ (resp. $\overline{X} = \overline{\xi}$) and using (i) and (ii) above, the result follows. □

In view of the above results together with Theorems 1.4 and 1.5, we have

Theorem 2.7. *If $\overline{\xi}(x) \in \mathfrak{X}(\pi(M))$ is a ∇-parallel π-vector field on (M, L). Then, we have*

(i) *$\overline{\xi}$ is D°-parallel.*
(ii) *$\overline{\xi}$ is D^{c}-parallel.*
(iii) *$\overline{\xi}$ is D^{*}-parallel.*

Theorem 2.8. *If $\overline{\xi}(x) \in \mathfrak{X}(\pi(M))$ is a ∇-parallel π-vector field on (M, L). Then, the curvature tensors associated with the Chern connection D^c, the Hashiguchi connection D^* and the Berwald connection D° have the same properties as Proposition 2.4.*

3. Special Finsler Spaces Admitting Parallel π-Vector Fields

In this section, we investigate the effect of the existence of a ∇-parallel (parallel) π-vector field which is independent of y on some important special Finsler spaces. The intrinsic definitions of the special Finsler spaces treated here are quoted from [14]. In what follows we assume that $\overline{\xi}(x) \neq 0$ is a parallel π-vector field independent of y.

For later use, we need the following lemma.

Lemma 3.1. *Let (M, L) be a Finsler manifold which admits a non-zero parallel π-vector $\overline{\xi}(x)$. Then, we have*:

(i) *The scalar function $B := g(\overline{\xi}, \overline{\eta})$ is everywhere non-zero.*

(ii) *The π-vector field $\overline{m} := \overline{\xi} - \frac{B}{L^2}\,\overline{\eta}$ is everywhere non-zero and is orthogonal to $\overline{\eta}$.*
(iii) *The π-vector fields $\overline{m}$ and $\overline{\xi}$ satisfy $g(\overline{m}, \overline{\xi}) = g(\overline{m}, \overline{m}) \neq 0$.*
(iv) *The angular metric tensor $\hbar$ satisfies $\hbar(\overline{\xi}, \overline{X}) \neq 0$ for all $\overline{X} \neq \overline{\eta}$.*

Proof. (i) If $B := g(\overline{\xi}, \overline{\eta}) = 0$, then

$$0 = (\nabla_{\gamma\overline{X}}\, g)(\overline{\xi}, \overline{\eta}) = \nabla_{\gamma\overline{X}}\, g(\overline{\xi}, \overline{\eta}) - g(\overline{\xi}, \overline{X}) = -g(\overline{\xi}, \overline{X}), \quad \forall\, \overline{X} \in \mathfrak{X}(\pi(M)),$$

which contradicts $\overline{\xi}(x) \neq 0$.

(ii) If $\overline{m} = 0$, then $L^2\overline{\xi} - B\overline{\eta} = 0$. Differentiating covariantly with respect to $\gamma\overline{X}$, we get

$$2g(\overline{X}, \overline{\eta})\overline{\xi} - B\overline{X} - g(\overline{X}, \overline{\xi})\overline{\eta} = 0. \tag{3.1}$$

From which,

$$g(\overline{X}, \overline{\xi}) = \frac{B}{L^2} g(\overline{X}, \overline{\eta}). \tag{3.2}$$

By (3.1), using (3.2), we obtain

$$\begin{aligned} 0 &= 2g(\overline{X}, \overline{\eta})g(\overline{Y}, \overline{\xi}) - Bg(\overline{X}, \overline{Y}) - g(\overline{X}, \overline{\xi})g(\overline{Y}, \overline{\eta}) \\ &= 2\frac{B}{L^2} g(\overline{Y}, \overline{\eta})g(\overline{X}, \overline{\eta}) - Bg(\overline{X}, \overline{Y}) - \frac{B}{L^2} g(\overline{X}, \overline{\eta})g(\overline{Y}, \overline{\eta}) \\ &= -B\left\{ g(\overline{X}, \overline{Y}) - \frac{1}{L^2} g(\overline{Y}, \overline{\eta})g(\overline{X}, \overline{\eta}) \right\} = -B\hbar(\overline{X}, \overline{Y}). \end{aligned}$$

From which, since $B \neq 0$, we are led to a contradiction: $\hbar = 0$.

On the other hand, the orthogonality of the two π-vector fields $\overline{m}$ and $\overline{\eta}$ follows from the identities $g(\overline{\eta}, \overline{\eta}) = L^2$ and $g(\overline{\eta}, \overline{\xi}) = B$.

(iii) Follows from (ii).

(iv) Suppose that $\hbar(\overline{X}, \overline{\xi}) = 0$ for all $\overline{X} \neq \overline{\eta} \in \mathfrak{X}(\mathcal{T}M)$, then, we have

$$0 = \hbar(\overline{X}, \overline{\xi}) = g(\overline{X}, \overline{\xi}) - \frac{1}{L^2} g(\overline{\xi}, \overline{\eta})g(\overline{X}, \overline{\eta}) = g(\overline{m}, \overline{X}),$$

which contradicts the fact that $\overline{m} \neq 0$. □

Definition 3.2. A Finsler manifold (M, L) is Riemannian if the metric tensor $g(x, y)$ is independent of y or, equivalently, if

$$T(\overline{X}, \overline{Y}) = 0, \quad \text{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

Definition 3.3. A Finsler manifold (M, L) is a Landsberg manifold if $\widehat{P}(\overline{X}, \overline{Y}) = 0$, or equivalently, if $\nabla_{\beta\overline{\eta}} T = 0$.

Definition 3.4. A Finsler manifold (M, L) is said to be:

(i) C^h-recurrent if the (h)hv-torsion tensor T satisfies the condition $\nabla_{\beta\overline{X}}\, T = \lambda_o(\overline{X})\, T$, where λ_o is a π-form of order one.

(ii) C^v-recurrent if the (h)hv-torsion tensor T satisfies the condition $(\nabla_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}) = \lambda_o(\overline{X})T(\overline{Y},\overline{Z})$.
(iii) C^0-recurrent if the (h)hv-torsion tensor T satisfies the condition $(D^\circ_{\gamma\overline{X}}T)(\overline{Y},\overline{Z}) = \lambda_o(\overline{X})T(\overline{Y},\overline{Z})$.

Theorem 3.5. *Let (M,L) be a Finsler manifold which admits a parallel π-vector field $\overline{\xi}(x)$ such that $\lambda_o(\overline{\xi}) \neq 0$. Then, the following assertions are equivalent*:

(i) *(M,L) is a C^h-recurrent manifold.*
(ii) *(M,L) is a C^v-recurrent manifold.*
(iii) *(M,L) is a C^0-recurrent manifold.*
(iv) *(M,L) is a Riemannian manifold.*

Proof. It is to be noted that (i), (ii) and (iv) are equivalent despite of the existence of a parallel π-vector field [14]. The implication (iv) $\Rightarrow$ (i) is trivial. It remains to prove that (i) $\Rightarrow$ (iv). Setting $\overline{W} = \overline{\xi}$ in (2.1), making use of $\widehat{P}(\overline{\xi},\overline{X}) = 0 = T(\overline{\xi},\overline{X})$ (Proposition 2.6), and $g((\nabla_{\beta\overline{Z}}T)(\overline{X},\overline{Y}),\overline{W}) = g((\nabla_{\beta\overline{Z}}T)(\overline{X},\overline{W}),\overline{Y})$ ([17, Proposition 3.3]), we get

$$\nabla_{\beta\overline{\xi}}T = 0.$$

On the other hand, Definition 3.4(i) for $\overline{X} = \overline{\xi}$, yields

$$\nabla_{\beta\overline{\xi}}T = \lambda_o(\overline{\xi})T.$$

The above two equations imply that $T = 0$ and hence (M,L) is Riemannian. □

Definition 3.6. A Finsler manifold (M,L) is said to be:

(i) Quasi-C-reducible if $\dim(M) \geq 3$ and the Cartan tensor T has the from

$$T(\overline{X},\overline{Y},\overline{Z}) = A(\overline{X},\overline{Y})C(\overline{Z}) + A(\overline{Y},\overline{Z})C(\overline{X}) + A(\overline{Z},\overline{X})C(\overline{Y}),$$

where A is a symmetric π-tensor field satisfying $A(\overline{X},\overline{\eta}) = 0$.
(ii) Semi-C-reducible if $\dim M \geq 3$ and the Cartan tensor T has the form

$$\begin{aligned} T(\overline{X},\overline{Y},\overline{Z}) = {} & \frac{\mu}{n+1}\{\hbar(\overline{X},\overline{Y})C(\overline{Z}) + \hbar(\overline{Y},\overline{Z})C(\overline{X}) + \hbar(\overline{Z},\overline{X})C(\overline{Y})\} \\ & + \frac{\tau}{C^2}C(\overline{X})C(\overline{Y})C(\overline{Z}), \end{aligned} \tag{3.3}$$

where $C^2 := C(\overline{C}) \neq 0$, μ and τ are scalar functions satisfying $\mu + \tau = 1$.
(iii) C-reducible if $\dim M \geq 3$ and the Cartan tensor T has the form

$$T(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{n+1}\{\hbar(\overline{X},\overline{Y})C(\overline{Z}) + \hbar(\overline{Y},\overline{Z})C(\overline{X}) + \hbar(\overline{Z},\overline{X})C(\overline{Y})\}. \tag{3.4}$$

(iv) C_2-like if $\dim M \geq 2$ and the Cartan tensor T has the form

$$T(\overline{X},\overline{Y},\overline{Z}) = \frac{1}{C^2}C(\overline{X})C(\overline{Y})C(\overline{Z}).$$

Proposition 3.7. *If a quasi-C-reducible Finsler manifold (M, L) $(\dim M \geq 3)$ admits a parallel π-vector field $\overline{\xi}(x)$, then (M, L) is Riemannian, provided that $A(\overline{\xi}, \overline{\xi}) \neq 0$.*

Proof. Follows from the defining property of quasi-C-reducibility by setting $\overline{X} = \overline{Y} = \overline{\xi}$ and using the fact that $C(\overline{\xi}) = 0$ and $A(\overline{\xi}, \overline{\xi}) \neq 0$. □

Theorem 3.8. *Let (M, L) be a Finsler manifold $(\dim M \geq 3)$ which admits a parallel π-vector field $\overline{\xi}(x)$, then, we have*

(i) *A C-reducible manifold (M, L) is a Riemannian manifold.*
(ii) *A semi-C-reducible manifold (M, L) is a C_2-like manifold.*

Proof. (i) Follows from the defining property of C-reducibility by setting $\overline{X} = \overline{Y} = \overline{\xi}$, taking into account Lemma 3.1(iv), Lemma 1.3 and $C(\overline{\xi}) = 0$.

(ii) Let (M, L) be semi-C-reducible. Setting $\overline{X} = \overline{Y} = \overline{\xi}$ and $\overline{Z} = \overline{C}$ in (3.3), taking into account Proposition 2.6(i) and $C(\overline{\xi}) = 0$, we get

$$\mu \hbar(\overline{\xi}, \overline{\xi}) C(\overline{C}) = 0.$$

From which, since $\hbar(\overline{\xi}, \overline{\xi}) \neq 0$ and $C(\overline{C}) \neq 0$, it follows that $\mu = 0$. Consequently, (M, L) is C_2-like. □

Definition 3.9. The condition

$$\mathbb{T}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) := L(\nabla_{\gamma \overline{X}} T)(\overline{Y}, \overline{Z}, \overline{W}) + \mathfrak{S}_{\overline{X}, \overline{Y}, \overline{Z}, \overline{W}} \, \ell(\overline{X}) T(\overline{Y}, \overline{Z}, \overline{W}) = 0, \quad (3.5)$$

will be called the $\mathbb{T}$-condition.

The more relaxed condition

$$\mathbb{T}_o(\overline{X}, \overline{Y}) := L(\nabla_{\gamma \overline{X}} C)(\overline{Y}) + \mathfrak{S}_{\overline{X}, \overline{Y}} \, \ell(\overline{X}) C(\overline{Y}) = 0. \quad (3.6)$$

will be called the $\mathbb{T}_o$-condition.

Theorem 3.10. *Let (M, L) be a Finsler manifold which admits a parallel π-vector field $\overline{\xi}(x)$. Then, the following assertions are equivalent*:

(i) *(M, L) satisfies the $\mathbb{T}$-condition.*
(ii) *(M, L) satisfies the $\mathbb{T}_o$-condition.*
(iii) *(M, L) is Riemannian.*

Proof. (i) $\Rightarrow$ (iii): Follows from (3.5) by setting $\overline{W} = \overline{\xi}$, taking into account that $T(\overline{X}, \overline{\xi}) = T(\overline{\xi}, \overline{X}) = 0$ and $\ell(\overline{\xi}) = \frac{B}{L} \neq 0$.

(ii) $\Rightarrow$ (iii): Follows from (3.6) by setting $\overline{Y} = \overline{\xi}$, taking into account the fact that $\ell(\overline{\xi}) \neq 0$.

The other implications are trivial. □

Definition 3.11. A Finsler manifold (M, L) is said to be S_3-like if $\dim(M) \geq 4$ and the v-curvature tensor S has the form:

$$S(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \frac{Sc^v}{(n-1)(n-2)}\{\hbar(\overline{X}, \overline{Z})\hbar(\overline{Y}, \overline{W}) - \hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{Z})\}. \quad (3.7)$$

Theorem 3.12. *If an S_3-like manifold (M, L) ($\dim M \geq 4$) admits a parallel π-vector field $\overline{\xi}(x)$, then, the v-curvature tensor S vanishes.*

Proof. Setting $\overline{Z} = \overline{\xi}$ in (3.7), taking Proposition 2.4 into account, we immediately get

$$\frac{Sc^v}{(n-1)(n-2)}\{\hbar(\overline{X}, \overline{\xi})\hbar(\overline{Y}, \overline{W}) - \hbar(\overline{X}, \overline{W})\hbar(\overline{Y}, \overline{\xi})\} = 0.$$

Taking the trace of the above equation, we have

$$\frac{Sc^v}{(n-1)(n-2)}\{(n-1)\hbar(\overline{X}, \overline{\xi}) - \hbar(\overline{X}, \overline{\xi})\} = \frac{Sc^v}{(n-1)}\hbar(\overline{X}, \overline{\xi}) = 0.$$

From which, since $\hbar(\overline{X}, \overline{\xi}) \neq 0$ (Lemma 3.1), the vertical scalar curvature Sc^v vanishes. Now, again, from (3.7), the result follows. □

Definition 3.13. A Finsler manifold (M, L), where $\dim M \geq 3$, is said to be:

(i) P_2-like if the hv-curvature tensor P has the form:

$$P(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \omega(\overline{Z})T(\overline{X}, \overline{Y}, \overline{W}) - \omega(\overline{W})\,T(\overline{X}, \overline{Y}, \overline{Z}), \quad (3.8)$$

where ω is a (1) π-form (positively homogeneous of degree 0).

(ii) P-reducible if the π-tensor field $\widehat{P}(\overline{X}, \overline{Y}, \overline{Z}) := g(\widehat{P}(\overline{X}, \overline{Y}), \overline{Z})$ has the form

$$\widehat{P}(\overline{X}, \overline{Y}, \overline{Z}) = \delta(\overline{X})\hbar(\overline{Y}, \overline{Z}) + \delta(\overline{Y})\hbar(\overline{X}, \overline{Z}) + \delta(\overline{Z})\hbar(\overline{X}, \overline{Y}), \quad (3.9)$$

where δ is the π-form defined by $\delta(\overline{X}) = \frac{1}{n+1}(\nabla_{\beta\overline{\eta}}C)(\overline{X})$.

Theorem 3.14. *Let (M, L) be a Finsler manifold ($\dim M \geq 3$) which admits a parallel π-vector field $\overline{\xi}(x)$, then, we have*

(i) *A P_2-like manifold (M, L) is a Riemannian manifold, provided that $\omega(\overline{\xi}) \neq -1$.*

(ii) *A P-reducible manifold (M, L) is a Landsberg manifold.*

Proof. (i) Setting $\overline{Z} = \overline{\xi}$ in (3.8), taking into account Propositions 2.4 and 2.6, we immediately get

$$(\omega(\overline{\xi}) + 1)T(\overline{X}, \overline{Y}) = 0.$$

Hence, the result follows.

(ii) Setting $\overline{X} = \overline{Y} = \overline{\xi}$ in (3.9) and taking into account that $(\nabla_{\beta\overline{\eta}}C)(\overline{\xi}) = 0$, we get $\hbar(\overline{\xi}, \overline{\xi})(\nabla_{\beta\overline{\eta}}C)(\overline{Z}) = 0$, with $\hbar(\overline{\xi}, \overline{\xi}) \neq 0$ (Lemma 3.1). Consequently, $\nabla_{\beta\overline{\eta}}C = 0$. Hence, again, from Definition 3.13(b), the (v)hv-torsion tensor $\widehat{P} = 0$. □

Definition 3.15. A Finsler manifold (M, L), where $\dim M \geq 3$, is said to be:

(i) h-isotropic if there exists a scalar k_o such that the horizontal curvature tensor R has the form

$$R(\overline{X}, \overline{Y})\overline{Z} = k_o\{g(\overline{X}, \overline{Z})\overline{Y} - g(\overline{Y}, \overline{Z})\overline{X}\};$$

(ii) of scalar curvature if there exists a scalar function $k : \mathcal{T}M \to \mathbb{R}$ such that

$$R(\overline{\eta}, \overline{X}, \overline{\eta}, \overline{Y}) = kL^2\hbar(\overline{X}, \overline{Y}),$$

where k is called the scalar curvature.

Theorem 3.16. *Let* (M, L), $\dim M \geq 3$, *be an* h*-isotropic Finsler manifold admitting a parallel* π*-vector field* $\overline{\xi}(x)$, *then the* h*-curvature tensor* R *of the Cartan connection vanishes.*

Proof. From Definition 3.15(i), we have

$$R(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = k_o\{g(\overline{X}, \overline{Z})g(\overline{Y}, \overline{W}) - g(\overline{Y}, \overline{Z})g(\overline{X}, \overline{W})\}. \tag{3.10}$$

Setting $\overline{Z} = \overline{\xi}$ and $\overline{X} = \overline{m}$ and noting that $R(\overline{X}, \overline{Y})\overline{\xi} = 0$ (Proposition 2.4), we have

$$k_o\{g(\overline{m}, \overline{\xi})g(\overline{Y}, \overline{W}) - g(\overline{Y}, \overline{\xi})g(\overline{m}, \overline{W})\} = 0.$$

Taking the trace of this equation, we get

$$k_o(n-1)g(\overline{m}, \overline{\xi}) = 0.$$

From which, since $g(\overline{m}, \overline{\xi}) = g(\overline{m}, \overline{m}) \neq 0$ (Lemma 3.1) and $\dim M \geq 3$, the scalar k_o vanishes. Now, again, from (3.10), the result follows. □

Theorem 3.17. *Let* (M, L), $\dim M \geq 3$, *be a Finsler manifold of scalar curvature admitting a parallel* π*-vector field* $\overline{\xi}(x)$. *The following statements hold*:

(i) *The scaler curvature* k *vanishes.*
(ii) *The deviation tensor field* H $(H(\overline{X}) := \widehat{R}(\overline{\eta}, \overline{X}))$ *vanishes.*
(iii) *The* (v)h*-torsion tensor* $\widehat{R}$ *of Cartan connection vanishes.*
(iv) *The* h*-curvature tensor* R° *of Berwald connection vanishes.*
(v) *The horizontal distribution is completely integrable.*

Proof. (i) Follows from Definition 3.15(ii) by setting $\overline{Y} = \overline{\xi}$, taking into account Proposition 2.4(vii) and Lemma 3.1(v).
(ii) Follows from (i) and Definition 3.15(ii).
(iii), (iv) and (v) Follow from [17, Theorem 4.7]. □

4. Energy β-Change and a Parallel π-Vector Field

In the present and the next sections we consider a perturbation, by a parallel π-vector field $\overline{\xi}(x)$, of the energy function $E = \frac{1}{2}L^2$ of a Finsler structure L.

Let (M, L) be a Finsler manifold. Consider the change

$$\widetilde{L}^2(x,y) = L^2(x,y) + B^2(x,y), \quad \text{with } B := \alpha(\overline{\eta}) := g(\overline{\eta}, \overline{\xi}), \tag{4.1}$$

$\widetilde{L}$ defines a new Finsler structure on M. The Finsler structure $\widetilde{L}$ is said to be obtained from the Finsler structure L by the β-change (4.1). The β-change (4.1) will be referred to as an **energy β-change** (as it can be written in the form $\widetilde{E} = E + \frac{1}{2}B^2$, where E and $\widetilde{E}$ are the energy functions corresponding to the Lagrangians L and $\widetilde{L}$ respectively).

The following two lemmas are useful for subsequent use.

Lemma 4.1. *The function $B(x,y)$ given by (4.1) has the properties*

(i) $B = d_J E(\beta\overline{\xi}), \quad d_J B = \alpha \circ \rho, \quad d_h B = 0.$
(ii) $dd_J B^2(\gamma\overline{X}, \beta\overline{Y}) = 2\alpha(\overline{X})\alpha(\overline{Y}), \quad dd_J B^2(\gamma\overline{X}, \gamma\overline{Y}) = 0.$
(iii) $dd_J B^2(\beta\overline{X}, \beta\overline{Y}) = 0.$

Proof. (i) From $g(\overline{\eta}, \overline{\eta}) = 2E^2$ and $\nabla g = 0$, one can show that

$$d_J E(X) = g(\rho X, \overline{\eta}), \quad \forall\, X \in \mathfrak{X}(\mathcal{T}M).$$

Setting $X = \beta\overline{\xi}$, we obtain $B = d_J E(\beta\overline{\xi})$.

On the other hand,

$$d_J B(X) = JX \cdot B = JX \cdot g(\overline{\xi}, \overline{\eta}) = g(\overline{\xi}, \nabla_{JX}\overline{\eta}) = g(\overline{\xi}, \rho X) = \alpha(\rho X).$$

Similarly,

$$d_h B(X) = hX \cdot B = hX \cdot g(\overline{\xi}, \overline{\eta}) = g(\nabla_{hX}\,\overline{\xi}, \overline{\eta}) + g(\overline{\xi}, \nabla_{hX}\,\overline{\eta}) = 0.$$

(ii) Making use of (i), we have

$$\begin{aligned} dd_J B^2(\gamma\overline{X}, \beta\overline{Y}) &= \gamma\overline{X} \cdot d_J B^2(\beta\overline{Y}) - \beta\overline{Y} \cdot d_J B^2(\gamma\overline{X}) - d_J B^2([\gamma\overline{X}, \beta\overline{Y}]) \\ &= 2\gamma\overline{X} \cdot (B\gamma\overline{Y} \cdot B) - 2\beta\overline{Y} \cdot (BJ\gamma\overline{X} \cdot B) - 2BJ[\gamma\overline{X}, \beta\overline{Y}] \cdot B \\ &= 2\gamma\overline{X} \cdot (Bg(\overline{Y}, \overline{\xi})) - 2Bg(\rho[\gamma\overline{X}, \beta\overline{Y}], \overline{\xi}) \\ &= 2\left\{g(\overline{X}, \overline{\xi})g(\overline{Y}, \overline{\xi}) + B\gamma\overline{X} \cdot g(\overline{Y}, \overline{\xi})\right\} \\ &\quad - 2Bg(\nabla_{\gamma\overline{X}}\overline{Y} - T(\overline{X}, \overline{Y}), \overline{\xi}) \\ &= 2\{g(\overline{X}, \overline{\xi})g(\overline{Y}, \overline{\xi}) + Bg(\nabla_{\gamma\overline{X}}\overline{Y}, \overline{\xi})\} - 2Bg(\nabla_{\gamma\overline{X}}\overline{Y}, \overline{\xi}) \\ &= 2\alpha(\overline{X})\alpha(\overline{Y}). \end{aligned}$$

Similarly, $dd_J B^2(\gamma\overline{X}, \gamma\overline{Y}) = 0$.

(iii) The proof is analogous to that of (ii). $\square$

The following lemma establishes some link between the pullback approach and Klein–Grifone approach and useful for subsequent use.

Lemma 4.2. *Let (M, L) be a Finsler manifold. Let g be the Finsler metric associated with L and ∇ the Cartan connection on $\pi^{-1}(TM)$. Then, the following relation holds*

$$g(\overline{X}, \overline{Y}) = \Omega(\gamma\overline{X}, \beta\overline{Y}) \quad \text{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)), \tag{4.2}$$

where $\Omega := dd_J E$ and β is the connection map associated with ∇.

The following result gives the relationship between g and $\widetilde{g}$.

Proposition 4.3. *Under the energy β-change (4.1), the Finsler metrics g and $\widetilde{g}$ are related by*

$$\widetilde{g}(\overline{X}, \overline{Y}) = g(\overline{X}, \overline{Y}) + \alpha(\overline{X})\alpha(\overline{Y}), \quad \text{for all } \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)), \tag{4.3}$$

α being the π-form associated with $\overline{\xi}$ under the duality defined by the metric g.

Proof. The proof follows by applying the operator $\frac{1}{2}dd_J$ on both sides of (4.1), taking into account (4.2), Lemma 4.2 and Lemma 4.1. □

Theorem 4.4. *Let (M, L) and $(M, \tilde{L})$ be two Finsler manifolds related by 4.1. The associated Barthel connections $\tilde{\Gamma}$ and Γ are related by*

$$\tilde{\Gamma} = \Gamma. \tag{4.4}$$

Consequently, $\tilde{h} = h$, $\tilde{v} = v$, or equivalently, $\tilde{\beta} = \beta$, $\tilde{K} = K$.

Proof. Since $\tilde{L}^2(x, y) = L^2(x, y) + B^2(x, y)$, then, using the fact that $E = \frac{1}{2}L^2$, we get

$$\tilde{E} = E + \frac{1}{2}B^2.$$

From which and using that $\Omega := dd_J E$, we obtain

$$\tilde{\Omega} = \Omega + \frac{1}{2}dd_J B^2.$$

As the difference between two sprays is a vertical vector field, assume that $\widetilde{G} = G + \gamma\overline{\mu}$, for some $\overline{\mu} \in \mathfrak{X}(\pi(M))$, then we have

$$\begin{aligned} -d\widetilde{E}(X) &= i_{\widetilde{G}}\,\widetilde{\Omega}(X) = i_{G+\gamma\overline{\mu}}\,\widetilde{\Omega}(X) \\ &= i_G\Omega(X) + \frac{1}{2}i_G\, dd_J B^2(X) + i_{\gamma\overline{\mu}}\,\Omega(X) + \frac{1}{2}i_{\gamma\overline{\mu}}\, dd_J B^2(X). \end{aligned} \tag{4.5}$$

Now, we compute the terms on the right-hand side (using Lemmas 4.1 and 4.2):

$$\begin{aligned} i_G\,\Omega(X) &= -dE(X). \\ i_{\gamma\overline{\mu}}\,\Omega(X) &= \Omega(\gamma\overline{\mu}, X) = \Omega(\gamma\overline{\mu}, \gamma KX + \beta\rho X) = \Omega(\gamma\overline{\mu}, \beta\rho X) = g(\overline{\mu}, \rho X). \end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\, i_G\, dd_J B^2(X) &= \frac{1}{2}\{dd_J B^2(\beta\overline{\eta}, \beta\rho X) - dd_J B^2(\gamma KX, \beta\overline{\eta})\} \\
&= -g(KX, \overline{\xi})g(\overline{\eta}, \overline{\xi}) = -X \cdot g(\overline{\eta}, \overline{\xi})g(\overline{\eta}, \overline{\xi}) \\
&= -BdB(X). \\
\frac{1}{2}\, i_{\gamma\overline{\mu}}\, dd_J B^2(X) &= \frac{1}{2}\, dd_J B^2(\gamma\overline{\mu}, \beta\rho X + \gamma KX) \\
&= \frac{1}{2}\, dd_J B^2(\gamma\overline{\mu}, \beta\rho X) = g(\overline{\xi}, \overline{\mu})g(\overline{\xi}, \rho X).
\end{aligned}$$

From these, together with $d\widetilde{E}(X) = dE(X) + BdB(X)$, Eq. (4.5) reduces to

$$g(\overline{\mu}, \rho X) + g(\overline{\xi}, \overline{\mu})g(\overline{\xi}, \rho X) = 0, \tag{4.6}$$

from which by setting $X = \beta\overline{\xi}$, we obtain

$$g(\overline{\xi}, \overline{\mu}) = 0. \tag{4.7}$$

From Eqs. (4.6) and (4.7), we have $\overline{\mu} = 0$, and consequently

$$\tilde{G} = G. \tag{4.8}$$

Now, by using $\Gamma = [J, G]$, together with (4.8), the result follows. □

We terminate this section by the following

Theorem 4.5. *Under energy β-change (4.1), the following geometric object are invariant*:

(i) *The canonical spray G.*
(ii) *The nonlinear connection $\Gamma := [J, G]$.*
(iii) *The curvature $\Re$ of the nonlinear connection Γ.*

Proof. The proof is clear and we omit it. □

5. Energy β-Change and Fundamental Linear Connections

In this section, we investigate the transformation of the fundamental linear connections of Finsler geometry, as well as their curvature tensors, under the energy β-change (4.1). We start our investigation with the Cartan connection.

The following two lemmas useful for subsequent use

Lemma 5.1 ([13]). *Let (M, L) be a Finsler manifold. Let g be the Finsler metric associated with L and let ∇ be the Cartan connection determined by the metric g. Then, the following relations hold*

(i) $2g(\nabla_{vX}\rho Y, \rho Z) = vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[Z, vX]) + g(\rho Z, \rho[vX, Y])$.
(ii) $2g(\nabla_{hX}\rho Y, \rho Z) = hX \cdot g(\rho Y, \rho Z) + hY \cdot g(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y)$
$\qquad - g(\rho X, \rho[hY, hZ]) + g(\rho Y, \rho[hZ, hX]) + g(\rho Z, \rho[hX, hY])$.

Since the Cartan connection ∇ and the Barthel connection $\Gamma =: [J, G]$ have the same horizontal and vertical projectors, then we have:

Lemma 5.2. *Under β-change given by (4.1), the Cartan horizontal projectors h and $\tilde{h}$ (resp. the Catan vertical projectors v and $\tilde{v}$) are related by $\tilde{h} = h, (resp.\ \tilde{v} = v)$.*

Theorem 5.3. *Let (M, L) and $(M, \tilde{L})$ be two Finsler manifolds are related by (4.1). Then the associated Cartan connections ∇ and $\tilde{\nabla}$ are related by:*

$$\tilde{\nabla}_X \rho Y = \nabla_X \rho Y, \quad \text{for all } X, Y \in \mathfrak{X}(\mathcal{T}M). \tag{5.1}$$

In particular,

(i) $\tilde{\nabla}_{\gamma\overline{X}}\overline{Y} = \nabla_{\gamma\overline{X}}\overline{Y}$,
(ii) $\tilde{\nabla}_{\beta\overline{X}}\overline{Y} = \nabla_{\beta\overline{X}}\overline{Y}$.

Proof. Using Lemma 5.1(i) and Proposition 4.3, taking into account Lemma 5.2, we have

$$\begin{aligned} 2\widetilde{g}(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \rho Z) &= \widetilde{v}X \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{g}(\rho Y, \rho[Z, \widetilde{v}X]) + \widetilde{g}(\rho Z, \rho[\widetilde{v}X, Y]) \\ &= vX \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{g}(\rho Y, \rho[hZ, vX]) + \widetilde{g}(\rho Z, \rho[vX, hY]) \\ &= vX \cdot \{g(\rho Y, \rho Z) + g(\rho Y, \overline{\xi})g(\rho Z, \overline{\xi})\} \\ &\quad + g(\rho Y, \rho[hZ, vX]) + g(\rho Y, \overline{\xi})g(\rho[hZ, vX], \overline{\xi}) \\ &\quad + g(\rho Z, \rho[vX, hY]) + g(\rho Z, \overline{\xi})g(\rho[vX, hY], \overline{\xi}) \\ &= \{vX \cdot g(\rho Y, \rho Z) + g(\rho Y, \rho[hZ, vX]) + g(\rho Z, \rho[vX, hY])\} \\ &\quad + \{g(\nabla_{vX}\rho Y, \overline{\xi})g(\rho Z, \overline{\xi}) + g(\rho Y, \overline{\xi})g\left(\nabla_{vX}\rho Z, \overline{\xi}\right) \\ &\quad + g(\rho Y, \overline{\xi})g(\mathbf{T}(vX, hZ) - \nabla_{vX}\rho Z, \overline{\xi}) \\ &\quad + g(\rho Z, \overline{\xi})g\left(\nabla_{vX}\rho Y - \mathbf{T}(vX, hY), \overline{\xi}\right)\}. \end{aligned}$$

Since $g(\mathbf{T}(vX, hY), \overline{\xi}) = -g(T(KX, \overline{\xi}), \rho Y) = 0$, the above relation takes the form

$$2\tilde{g}(\tilde{\nabla}_{vX}\rho Y, \rho Z) = 2g(\nabla_{vX}\rho Y, \rho Z) + 2g(g(\nabla_{vX}\rho Y, \overline{\xi})\overline{\xi}, \rho Z).$$

Consequently,

$$\begin{aligned} g(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \rho Z) + g(g(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \overline{\xi})\,\overline{\xi}, \rho Z) &= g(\nabla_{vX}\rho Y, \rho Z) \\ &\quad + g(g(\nabla_{vX}\rho Y, \overline{\xi})\,\overline{\xi}, \rho Z). \end{aligned} \tag{5.2}$$

From which, by setting $Z = \beta\overline{\xi}$, we get

$$g(\widetilde{\nabla}_{\widetilde{v}X}\rho Y, \overline{\xi}) - g(\nabla_{vX}\rho Y, \overline{\xi}) = 0. \tag{5.3}$$

Then, Eqs. (5.2) and (5.3) imply that

$$\widetilde{\nabla}_{\widetilde{v}X}\rho Y = \nabla_{\widetilde{v}X}\rho Y. \tag{5.4}$$

Similarly, by Lemma 2.3(ii) and Proposition 4.2, taking into account Lemma 5.2 and the fact that $T(\overline{\xi}, \overline{X}) = 0 = T(\overline{X}, \overline{\xi})$, we get after long but easy calculations

$$\begin{aligned}
2\widetilde{g}(\widetilde{\nabla}_{\widetilde{h}X}\rho Y, \rho Z) &= \widetilde{h}X \cdot \widetilde{g}(\rho Y, \rho Z) + \widetilde{h}Y \cdot \widetilde{g}(\rho Z, \rho X) - \widetilde{h}Z \cdot g(\rho X, \rho Y) \\
&\quad - \widetilde{g}(\rho X, \rho[\widetilde{h}Y, \widetilde{h}Z]) + \widetilde{g}(\rho Y, \rho[\widetilde{h}Z, \widetilde{h}X]) + \widetilde{g}(\rho Z, \rho[\widetilde{h}X, \widetilde{h}Y]) \\
&= hX \cdot \widetilde{g}(\rho Y, \rho Z) + hY \cdot \widetilde{g}(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y) \\
&\quad - \widetilde{g}(\rho X, \rho[hY, hZ]) + \widetilde{g}(\rho Y, \rho[hZ, hX]) + \widetilde{g}(\rho Z, \rho[hX, hY]) \\
&= hX \cdot \{g(\rho Y, \rho Z) + g(\rho Y, \overline{\xi})g(\rho Z, \overline{\xi})\} \\
&\quad + hY \cdot \{g(\rho Z, \rho X) + g(\rho Z, \overline{\xi})g(\rho X, \overline{\xi})\} \\
&\quad - hZ \cdot \{g(\rho X, \rho Y) + g(\rho X, \overline{\xi})g(\rho Y, \overline{\xi})\} \\
&\quad - \{g(\rho X, \rho[hY, hZ]) + g(\rho X, \overline{\xi})g(\rho[hY, hZ], \overline{\xi})\} \\
&\quad + \{g(\rho Y, \rho[hZ, hX]) + g(\rho Y, \overline{\xi})g(\rho[hZ, hX], \overline{\xi})\} \\
&\quad + \{g(\rho Z, \rho[hX, hY]) + g(\rho Z, \overline{\xi})g(\rho[hX, hY], \overline{\xi})\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
2\widetilde{g}(\widetilde{\nabla}_{\widetilde{h}X}\rho Y, \rho Z) &= \{hX \cdot g(\rho Y, \rho Z) + hY \cdot g(\rho Z, \rho X) - hZ \cdot g(\rho X, \rho Y) \\
&\quad - g(\rho X, \rho[hY, hZ]) + g(\rho Y, \rho[hZ, hX]) + g(\rho Z, \rho[hX, hY])\} \\
&\quad + \{g(\nabla_{hX}\rho Y, \overline{\xi})g(\rho Z, \overline{\xi}) + g(\nabla_{hX}\rho Z, \overline{\xi})g(\rho Y, \overline{\xi})\} \\
&\quad + \{g(\nabla_{hY}\rho Z, \overline{\xi})g(\rho X, \overline{\xi}) + g(\nabla_{hY}\rho X, \overline{\xi})g(\rho Z, \overline{\xi})\} \\
&\quad - \{g(\nabla_{hZ}\rho X, \overline{\xi})g(\rho Y, \overline{\xi}) + g(\nabla_{hZ}\rho Y, \overline{\xi})g(\rho X, \overline{\xi})\} \\
&\quad - \{g(\rho X, \overline{\xi})g(\nabla_{hY}\rho Z - \nabla_{hZ}\rho Y, \overline{\xi})\} \\
&\quad + \{g(\rho Y, \overline{\xi})g(\nabla_{hZ}\rho X - \nabla_{hX}\rho Z, \overline{\xi})\} \\
&\quad + \{g(\rho Z, \overline{\xi})g(\nabla_{hX}\rho Y - \nabla_{hY}\rho X, \overline{\xi})\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
g(\widetilde{\nabla}_{\widetilde{h}X}\rho Y, \rho Z) + g(g(\widetilde{\nabla}_{\widetilde{h}X}\rho Y, \overline{\xi})\,\overline{\xi}, \rho Z) &= g(\nabla_{hX}\rho Y, \rho Z) \\
&\quad + g(g(\nabla_{hX}\rho Y, \overline{\xi})\,\overline{\xi}, \rho Z). \qquad (5.5)
\end{aligned}$$

From which, by setting $Z = \beta\overline{\xi}$ and taking into account $\beta o \rho = \mathrm{id}_{\mathfrak{X}(\pi(M))}$, we have

$$\tilde{g}(\tilde{\nabla}_{hX}\rho Y, \overline{\xi}) - g(\nabla_{hX}\rho Y, \overline{\xi}) = 0. \qquad (5.6)$$

Now, from Eqs. (5.5) and (5.6), taking into account the fact that the Finsler metric tensor g is nondegenerate, we obtain

$$\tilde{\nabla}_{hX}\rho Y = \nabla_{hX}\rho Y. \qquad (5.7)$$

Therefore, by (5.4) and (5.7), taking into account the fact that $h + v = I$, the Eq. (5.1) follows.

Finally, Part (a) and (b) follows from (5.1) by setting $X = \gamma\overline{X}$ (resp. $X = \beta\overline{X}$) and $Y = \beta\overline{Y}$, noting that $\rho\, o\, \gamma = 0$. □

As a consequence of Theorem 5.3 and Definition 2.1 yield

Proposition 5.4. *If a Finsler manifold with fundamental function L admits a parallel π-vector field $\overline{\xi}(x)$ with respect to ∇, then the vector field $\overline{\xi}$ is a parallel π-vector field with respect to the modified metric* (4.1).

In view of the above theorem and Lemma 5.2, we have

Corollary 5.5. *Under energy β-change* (4.1), *the following geometric objects are invariant*

(i) *The Cartan connection*: ∇.
(ii) *The torsion tensors of Cartan connection*: $T, \widehat{P}$ *and* $\widehat{R}$.
(iii) *The curvature tensors of Cartan connection*: S, P *and* R.
(iv) *The vertical and horizontal Ricci tensors of Cartan connection*: Ric^v *and* Ric^h.
(v) *The vertical and horizontal scaler functions of Cartan connection*: Sc^v *and* Sc^h.

We terminate our study by the energy β-change of Berwald connection D°, Chern connection D^c and Hashiguchi connection D^*.

From Theorems 1.4, 1.5, 5.3 and Lemma 5.2, taking into account the fact that the torsion tensors T and $\widehat{P}$ are invariant under the energy β-change (4.1) (Corollary 5.5(ii)), we have

Theorem 5.6. *Under energy β-change* (4.1), *the following geometric objects are invariant*

(i) *The Berwald connection*: D°.
(ii) *The torsion and curvature tensors of Berwald connection*: $\widehat{R}^\circ$ *and* P°, R°.
(iii) *The Chern connection*: D^c.
(iv) *The torsion and curvature tensors of Chern connection*: $\widehat{P}^c, \widehat{R}^c$ *and* P^c, R^c.
(v) *The Hashiguchi connection*: D^*.
(vi) *The torsion and curvature tensors of Hashiguchi connection*: $\widehat{R}^*$ *and* S^*, P^*, R^*.

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